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# Iterative methods for infinite families of nonexpansive mappings in Banach spaces

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## 1 Introduction

Throughout this paper, let  $E$  be a real Banach space with norm  $\|\cdot\|$  and let  $\mathbb{N}$  be the set of all positive integers. Let  $C$  be a nonempty closed convex subset of  $E$ . Then, a mapping  $T : C \rightarrow C$  is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all  $x, y \in C$ . Browder [4] considered a sequence  $\{x_n\}$  as follows:

$$x \in C, \quad x_n = \alpha_n x + (1 - \alpha_n)Tx_n \quad (\forall n \in \mathbb{N}), \quad (1.1)$$

where  $\{\alpha_n\} \subset (0, 1)$  and he proved the first strong convergence theorem in the framework of a Hilbert space. Later, Reich [29], Takahashi and Ueda [51], Shioji and Takahashi [39], Nakajo [21] and others also proved strong convergence theorems of Browder's type in Hilbert spaces or Banach spaces. On the other hand, Halpern [9] considered the following process:  $x_1 = x \in C$  and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \quad (\forall n \in \mathbb{N}), \quad (1.2)$$

where  $\{\alpha_n\} \subset [0, 1)$ . Wittmann [52] proved a strong convergence theorem of Halpern's type in the framework of a Hilbert space and then, several authors [2, 10, 11, 12, 13, 14, 17, 21, 33, 35, 38, 39, 40, 50] proved strong convergence theorems of Halpern's type in Hilbert spaces or Banach spaces. Recently, Moudafi [20] and Xu [53] considered the following process by the viscosity approximation method:  $x_1 = x \in C$  and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n \quad (\forall n \in \mathbb{N}), \quad (1.3)$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $f : C \rightarrow C$  is a contraction.

In this article, for an infinite family  $\{T_n\}$  of nonexpansive mappings of  $C$  into itself such that  $\emptyset \neq \bigcap_{n=1}^{\infty} F(T_n)$ , we consider a sequence  $\{x_n\}$  generated by

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $f : C \rightarrow C$  is a contraction. Then, we give the conditions of  $\{\alpha_n\}$  and  $\{T_n\}$  under which  $\{x_n\}$  converges strongly to a common fixed point of  $\bigcap_{n=1}^{\infty} F(T_n)$ . We also consider a sequence  $\{x_n\}$  generated by

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $f : C \rightarrow C$  is a contraction. Then, we give the conditions of  $\{\alpha_n\}$  and  $\{T_n\}$  under which  $\{x_n\}$  converges strongly to a common fixed point of  $\bigcap_{n=1}^{\infty} F(T_n)$ . Using these results, we improve and extend well-known strong convergence theorems.

## 2 Preliminaries

Let  $E$  be a real Banach space with norm  $\|\cdot\|$  and let  $E^*$  denote the dual of  $E$ . We denote the value of  $y^* \in E^*$  at  $x \in E$  by  $\langle x, y^* \rangle$ . The duality mapping  $J$  from  $E$  into  $2^{E^*}$  is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every  $x \in E$ . Let  $U = \{x \in E : \|x\| = 1\}$ . The norm of  $E$  is said to be Gâteaux differentiable if for each  $x, y \in U$ , the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists. In the case,  $E$  is called smooth. The norm of  $E$  is said to be uniformly Gâteaux differentiable if for each  $y \in U$ , the limit (2.1) is attained uniformly for  $x \in U$ . We know that if  $E$  is smooth, then the duality mapping  $J$  is single valued. Further, if the norm of  $E$  is uniformly Gâteaux differentiable, then  $J$  is norm to weak\* uniformly continuous on each bounded subset of  $E$ . Let  $C$  be a closed convex subset of  $E$ . A mapping  $T : C \rightarrow C$  is said to be nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|$  for all  $x, y \in C$ . We denote by  $F(T)$  the set of all fixed points of  $T$ . Let  $I$  denote the identity operator on  $E$ . An operator  $A \subset E \times E$  with domain  $D(A) = \{x \in E : Ax \neq \emptyset\}$  and range  $R(A) = \bigcup \{Ax : x \in D(A)\}$  is said to be accretive if for each  $x_i \in D(A)$  and  $y_i \in Ax_i, i = 1, 2$ , there exists  $j \in J(x_1 - x_2)$  such that  $\langle y_1 - y_2, j \rangle \geq 0$ . If  $A$  is accretive, then we have

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(y_1 - y_2)\|$$

for all  $r > 0$  and  $y_i \in Ax_i, i = 1, 2$ . If  $A$  is accretive, then we can define, for each  $r > 0$ , a nonexpansive single valued mapping  $J_r : R(I + rA) \rightarrow D(A)$  by  $J_r = (I + rA)^{-1}$ . It is called the resolvent of  $A$ . We also define the Yosida approximation  $A_r$  by  $A_r = (I - J_r)/r$ . We know that  $A_r x \in AJ_r x$  for all  $x \in R(I + rA)$  and  $\|A_r x\| \leq \inf\{\|y\| : y \in Ax\}$  for all  $x \in D(A) \cap R(I + rA)$ . We also know that for an accretive operator  $A$ ,  $A^{-1}0 = F(J_r)$  for all  $r > 0$ , where  $A^{-1}0 = \{u \in E : 0 \in Au\}$ . An accretive operator  $A$  is said to be  $m$ -accretive if  $R(I + rA) = E$  for all  $r > 0$ . A closed convex subset  $C$  of a Banach space  $E$  is said to have normal structure if for each bounded closed convex subset of  $K$  of  $C$  which contains at least two points, there exists an element  $x$  of  $K$  which is not a diametral point of  $K$ , i.e.,

$$\sup\{\|x - y\| : y \in K\} < \delta(K),$$

where  $\delta(K)$  is the diameter of  $K$ . It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure; see [44] for more details. The following result was proved by Kirk [18].

**Theorem 2.1 (Kirk [18]).** *Let  $E$  be a reflexive Banach space and let  $C$  be a nonempty bounded closed convex subset of  $E$  which has normal structure. Let  $T$  be a nonexpansive mapping of  $C$  into itself. Then  $F(T)$  is nonempty.*

A closed convex subset  $C$  of a Banach space  $E$  is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a nonempty bounded closed convex subset of  $C$  into itself has a fixed point in  $C$ . If  $C$  is a closed convex subset of a reflexive Banach space which has normal structure, from Theorem 2.1,  $C$  has the fixed point property for nonexpansive mappings.

We denote by  $\mathbb{N}$  the set of all natural numbers and let  $\mu$  be a mean on  $\mathbb{N}$ , i.e., a continuous linear functional on  $\ell^\infty$  satisfying  $\|\mu\| = 1 = \mu(1)$ . We know that  $\mu$  is a mean on  $\mathbb{N}$  if and only if

$$\inf_{n \in \mathbb{N}} a_n \leq \mu(f) \leq \sup_{n \in \mathbb{N}} a_n$$

for each  $f = (a_1, a_2, \dots) \in \ell^\infty$ . Occasionally, we use  $\mu_n(a_n)$  instead of  $\mu(f)$ . Let  $f = (a_1, a_2, \dots) \in \ell^\infty$  with  $a_n \rightarrow a$  and let  $\mu$  be a Banach limit on  $\mathbb{N}$ . Then,  $\mu(f) = \mu_n(a_n) = a$ ; see [44] for more details. Further, we know the following result [51].

**Theorem 2.2 (Takahashi and Ueda [51]).** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  with a uniformly Gâteaux differentiable norm, let  $\{x_n\}$  be a bounded sequence of  $E$  and let  $\mu$  be a mean on  $\mathbb{N}$ . Let  $z \in C$ . Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

*if and only if  $\mu_n \langle y - z, J(x_n - z) \rangle \leq 0$  for all  $y \in C$ , where  $J$  is the duality mapping of  $E$ .*

Let  $C$  be a nonempty subset of a Banach space  $E$ . Let  $D$  be a subset of  $C$  and let  $P$  be a retraction of  $C$  onto  $D$ , i.e.,  $Px = x$  for each  $x \in D$ . Then  $P$  is said to be sunny [28] if for each  $x \in C$  and  $t \geq 0$  with  $Px + t(x - Px) \in C$ ,

$$P(Px + t(x - Px)) = Px.$$

A subset  $D$  of  $C$  is said to be a sunny nonexpansive retract of  $C$  if there exists a sunny nonexpansive retraction  $P$  of  $C$  onto  $D$ . We know that if  $E$  is smooth and  $P$  is a retraction of  $C$  onto  $D$ , then  $P$  is sunny and nonexpansive if and only if for each  $x \in C$  and  $z \in D$ ,

$$\langle x - Px, J(z - Px) \rangle \leq 0. \quad (2.2)$$

For more details, see [44].

### 3 Conditions for infinite families

Let  $E$  be a Banach space and let  $C$  be a nonempty closed convex subset of  $E$ . Let  $\{T_n\}$  and  $\mathcal{T}$  be families of nonexpansive mappings of  $C$  into itself such that  $\emptyset \neq F(\mathcal{T}) = \bigcap_{n=1}^{\infty} F(T_n)$ , where  $F(T_n)$  is the set of all fixed points of  $T_n$  and  $F(\mathcal{T})$  is the set of all common fixed points of  $\mathcal{T}$ . Then,  $\{T_n\}$  is said to satisfy the condition (I) with  $\mathcal{T}$  if for each bounded sequence  $\{z_n\}$  in  $C$ ,

$$\lim_{n \rightarrow \infty} \|z_n - T_n z_n\| = 0$$

implies that  $\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0$  for all  $T \in \mathcal{T}$ . In particular, if  $\mathcal{T} = \{T\}$ , i.e.,  $\mathcal{T}$  consists of one mapping  $T$ , then  $\{T_n\}$  is said to satisfy the condition (I) with  $T$ .  $\{T_n\}$  is said to satisfy the condition (II) if for each bounded sequence  $\{z_n\} \subset C$ ,

$$\lim_{n \rightarrow \infty} \|z_{n+1} - T_n z_n\| = 0$$

implies that  $\lim_{n \rightarrow \infty} \|z_n - T_m z_n\| = 0$  for all  $m \in \mathbb{N}$ .  $\{T_n\}$  is said to satisfy the condition (III) if for every bounded subset  $B$  of  $C$ ,

$$\sum_{n=1}^{\infty} \sup\{\|T_n x - T_{n+1} x\| : x \in B\} < \infty.$$

**Proposition 3.1.** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself with  $F(T) \neq \emptyset$ . Then,  $\{T_n\}$  with  $T_n = T$  for all  $n \in \mathbb{N}$  satisfies the condition (I) with  $T$  and the condition (III).*

*Proof.* Put  $T_n = T$  for all  $n \in \mathbb{N}$ . Then, it is obvious that  $\{T_n\}$  satisfies the condition (I) with  $T$  and the condition (III).  $\square$

**Theorem 3.2 ([24]).** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  and let  $S$  and  $T$  be nonexpansive mappings of  $C$  into itself such that  $F(S) \cap F(T) \neq \emptyset$ . Let  $\{\gamma_n\} \subset [a, b]$  for some  $a, b \in (0, 1)$  with  $a \leq b$ . Then,  $\{T_n\}$  with  $T_n = \gamma_n S + (1 - \gamma_n)T$  for all  $n \in \mathbb{N}$  satisfies the condition (I) with  $\frac{S+T}{2}$ . Further,  $\{T_n\}$  with  $T_n = \gamma_n S + (1 - \gamma_n)T$  for all  $n \in \mathbb{N}$  such that  $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$  satisfies the condition (I) with  $\frac{S+T}{2}$  and the condition (III).*

The following lemma is related to Edelstein and O'Brien [6, Theorem 1].

**Lemma 3.3 ([48]).** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself with  $F(T) \neq \emptyset$ . Let  $\{\beta_n\}$  be a sequence of real numbers with  $0 < a \leq \beta_n \leq b < 1$  and let  $B$  be a nonempty bounded subset of  $C$ . Define a mapping  $S_n$  of  $C$  into itself by*

$$S_n x = S(\beta_n)x = (1 - \beta_n)x + \beta_n T x \quad \text{for all } x \in C$$

*and put  $a_n = \sup_{x \in B} \|TS^n x - S^n x\|$  for all  $n \in \mathbb{N}$ , where  $S^n = S_n S_{n-1} \cdots S_1$ . Then,  $a_n \rightarrow 0$ . In particular, for any  $m \in \mathbb{N}$ ,*

$$\lim_{n \rightarrow \infty} \sup_{x \in B} \|S_m S^n x - S^n x\| = 0.$$

The following lemma was also proved by Takahashi [48].

**Lemma 3.4 ([48]).** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself with  $F(T) \neq \emptyset$ . For a nonempty bounded subset  $B$  of  $C$  and  $n \in \mathbb{N}$ , define a mapping  $f_n$  of  $[0, 1]^n$  into  $(-\infty, \infty)$  by*

$$f_n(\beta_n, \beta_{n-1}, \dots, \beta_1) = \sup_{x \in B} \|TU^n x - U^n x\|$$

*for all  $(\beta_n, \beta_{n-1}, \dots, \beta_1) \in [0, 1]^n$ , where  $U^n = S(\beta_n)S(\beta_{n-1}) \cdots S(\beta_1)$  and*

$$S(\beta_k)x = (1 - \beta_k)x + \beta_k T x$$

*for all  $x \in C$  and  $k \in \{1, 2, \dots, n\}$ . Then,  $f_n$  is continuous.*

Using Lemmas 3.3 and 3.4 we obtain the following theorem.

**Theorem 3.5 ([48]).** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  and let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T)$  is nonempty. For any  $n \in \mathbb{N}$  and  $\beta_n \in \mathbb{R}$  with  $0 < a \leq \beta_n \leq b < 1$ , define  $S_n : C \rightarrow C$  as follows:*

$$S_n x = (1 - \beta_n)x + \beta_n T x \quad \text{for all } x \in C.$$

*Then,  $\{S_n\}$  satisfies the condition (I) with  $T$  and the condition (II).*

We know the following lemma for resolvents of accretive operators; see [44].

**Lemma 3.6.** *Let  $E$  be a Banach space and let  $A \subset E \times E$  be an accretive operator. Let  $r, \lambda > 0$  and  $D(A) \subset R(I + \lambda A)$ . Then,*

$$\frac{1}{\lambda} \|J_r x - J_{\lambda} J_r x\| \leq \frac{1}{r} \|x - J_r x\|$$

*for every  $x \in R(I + rA)$ .*

Using Lemma 3.6, we also have the following theorem.

**Theorem 3.7 ([48]).** *Let  $C$  be a nonempty closed convex subset of a Banach space  $E$  and let  $A \subset E \times E$  be an accretive operator such that*

$$\overline{D(A)} \subset C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$$

*and  $A^{-1}0 \neq \emptyset$ . Let  $\{\lambda_n\}$  be a sequence of real numbers such that  $\lambda_n \in (0, \infty)$  and  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . Define  $S_n = J_{\lambda_n}$  for any  $n \in \mathbb{N}$ . Then,  $\{S_n\}$  satisfies the condition (I) with  $J_1$  and the condition (II), where  $J_1 = (I + A)^{-1}$ . Moreover,  $\{T_n\}$  with  $T_n = J_{\lambda_n}$  ( $\forall n \in \mathbb{N}$ ) such that  $\{\lambda_n\} \subset (0, \infty)$ ,  $\liminf_{n \rightarrow \infty} \lambda_n > 0$  and  $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$  satisfies the condition (I) with  $\{J_1\}$  and the condition (III).*

Let  $C$  be a nonempty closed convex subset of  $E$ . Let  $S_1, S_2, \dots$  be infinite nonexpansive mappings of  $C$  into itself and let  $\beta_1, \beta_2, \dots$  be real numbers such that  $0 \leq \beta_i \leq 1$  for every  $i \in \mathbb{N}$ . Then, for any  $n \in \mathbb{N}$ , Takahashi [43] (see also [34, 45, 49]) introduced a mapping  $W_n$  of  $C$  into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \beta_n S_n U_{n,n+1} + (1 - \beta_n) I, \\ U_{n,n-1} &= \beta_{n-1} S_{n-1} U_{n,n} + (1 - \beta_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \beta_k S_k U_{n,k+1} + (1 - \beta_k) I, \\ &\vdots \\ U_{n,2} &= \beta_2 S_2 U_{n,3} + (1 - \beta_2) I, \\ W_n &= U_{n,1} = \beta_1 S_1 U_{n,2} + (1 - \beta_1) I. \end{aligned}$$

Such a mapping  $W_n$  is called the  $W$ -mapping generated by  $S_n, S_{n-1}, \dots, S_1$  and  $\beta_n, \beta_{n-1}, \dots, \beta_1$ . We know that if  $E$  is strictly convex,  $\cap_{i=1}^n F(S_i) \neq \emptyset$ ,  $0 < \beta_i < 1$  for every  $i = 2, 3, \dots, n$  and  $0 < \beta_1 \leq 1$ , then,  $F(W_n) = \cap_{i=1}^n F(S_i)$ ; see [45, 49]. We also have

that if  $E$  is strictly convex,  $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$  and  $0 < \beta_i \leq b < 1$  for every  $i \in \mathbb{N}$  for some  $b \in (0, 1)$ , then,  $\lim_{n \rightarrow \infty} U_{n,k}x$  exists for every  $x \in C$  and  $k \in \mathbb{N}$ ; see [34]. So, we can define a mapping  $W$  of  $C$  into itself as follows:

$$Wx = \lim_{n \rightarrow \infty} W_n x = \lim_{n \rightarrow \infty} U_{n,1}x$$

for every  $x \in C$ . Such a  $W$  is called the  $W$ -mapping generated by  $S_1, S_2, \dots$  and  $\beta_1, \beta_2, \dots$ . We have that if  $E$  is strictly convex,  $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$  and  $0 < \beta_i \leq b < 1$  for every  $i \in \mathbb{N}$  for some  $b \in (0, 1)$ , then,  $F(W) = \cap_{i=1}^{\infty} F(S_i)$ ; see [34]. We know the following result for the  $W$ -mappings.

**Theorem 3.8 ([24]).** *Let  $C$  be a nonempty closed convex subset of a strictly convex Banach space  $E$ . Let  $S_1, S_2, \dots$  be infinite nonexpansive mappings of  $C$  into itself with  $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$  and let  $\beta_1, \beta_2, \dots$  be real numbers with  $0 < \beta_i \leq b < 1$  for every  $i \in \mathbb{N}$  for some  $b \in (0, 1)$ . Let  $W_n$  be the  $W$ -mapping generated by  $S_n, S_{n-1}, \dots, S_1$  and  $\beta_n, \beta_{n-1}, \dots, \beta_1$  for every  $n \in \mathbb{N}$  and let  $W$  be the  $W$ -mapping generated by  $S_1, S_2, \dots$  and  $\beta_1, \beta_2, \dots$ . Then,  $\{T_n\}$  with  $T_n = W_n$  ( $\forall n \in \mathbb{N}$ ) satisfies the condition (I) with  $W$  and the condition (III).*

## 4 Strong convergence theorem of Browder's type

We can prove a strong convergence theorem of Browder's type for a countable family of nonexpansive mappings in a Banach space.

**Theorem 4.1 ([48]).** *Let  $E$  be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let  $C$  be a nonempty closed convex subset of  $E$  which has the fixed point property for nonexpansive mappings. Let  $T$  be a nonexpansive mapping of  $C$  into itself and let  $\{T_n\}$  be a family of nonexpansive mappings of  $C$  into itself which satisfies  $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ . Further, suppose that  $\{T_n\}$  satisfies the condition (I) with  $T$ . Define a sequence  $\{x_n\}$  in  $C$  as follows:*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) T_n x_n, \quad n = 1, 2, 3, \dots,$$

where  $\{\alpha_n\} \subset (0, 1)$  satisfies  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $f$  is a contraction of  $C$  into itself. Then,  $\{x_n\}$  converges strongly to  $u \in F(T)$ , where  $u = P_{F(T)} f(u)$  and  $P_{F(T)}$  is a sunny nonexpansive retraction of  $C$  onto  $F(T)$ .

We have the following result for nonexpansive mappings by Proposition 3.1 and Theorem 4.1.

**Theorem 4.2.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $T$  be a nonexpansive mapping of  $C$  into itself such that  $F(T) \neq \emptyset$ . Let  $x \in C$  and  $\{x_n\}$  be a sequence by  $x_n = \alpha_n x + (1 - \alpha_n) T_n x_n$  ( $\forall n \in \mathbb{N}$ ), where  $\{\alpha_n\} \subset (0, 1)$  with  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Then,  $\{x_n\}$  converges strongly to  $P_{F(T)} x$ , where  $P_{F(T)}$  is a sunny nonexpansive retraction of  $C$  onto  $F(T)$ .*

We also get the following result for nonexpansive mappings by Theorems 3.2 and 4.1.

**Theorem 4.3.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $S$  and  $T$  be nonexpansive mappings of  $C$  into itself such that  $F(S) \cap F(T) \neq \emptyset$ . Let  $x \in C$  and  $\{x_n\}$  be a sequence by  $x_n = \alpha_n x + (1 - \alpha_n)(\gamma_n S x_n + (1 - \gamma_n) T x_n)$  ( $\forall n \in \mathbb{N}$ ), where  $\{\alpha_n\} \subset (0, 1)$  with  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\{\gamma_n\} \subset [a, b]$  for some  $a, b \in (0, 1)$  with  $a \leq b$ . Then,  $\{x_n\}$  converges strongly to  $P_{F(S) \cap F(T)} x$ , where  $P_{F(S) \cap F(T)}$  is a sunny nonexpansive retraction of  $C$  onto  $F(S) \cap F(T)$ .*

We have the following result for accretive operators from Theorems 3.7 and 4.1.

**Theorem 4.4.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $A \subset E \times E$  be an accretive operator with  $\overline{D(A)} \subset C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$  and  $A^{-1}0 \neq \emptyset$ . Let  $x \in C$  and  $\{x_n\}$  be a sequence by  $x_n = \alpha_n x + (1 - \alpha_n)J_{\lambda_n} x_n$  ( $\forall n \in \mathbb{N}$ ), where  $\{\lambda_n\} \subset (0, \infty)$  and  $\{\alpha_n\} \subset (0, 1)$  with  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . If  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ ,  $\{x_n\}$  converges strongly to  $P_{A^{-1}0}x$ , where  $P_{A^{-1}0}$  is a sunny nonexpansive retraction of  $C$  onto  $A^{-1}0$ .*

We get the following result for the  $W$ -mappings from Theorems 3.8 and 4.1.

**Theorem 4.5.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable. Let  $S_1, S_2, \dots$  be infinite nonexpansive mappings of  $C$  into itself with  $F := \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$  and let  $\beta_1, \beta_2, \dots$  be real numbers with  $0 < \beta_i \leq b < 1$  for every  $i \in \mathbb{N}$  for some  $b \in (0, 1)$ . Let  $W_n$  be the  $W$ -mapping generated by  $S_n, S_{n-1}, \dots, S_1$  and  $\beta_n, \beta_{n-1}, \dots, \beta_1$  for every  $n \in \mathbb{N}$ . Let  $x \in C$  and  $\{x_n\}$  be a sequence by  $x_n = \alpha_n x + (1 - \alpha_n)W_n x_n$  ( $\forall n \in \mathbb{N}$ ), where  $\{\alpha_n\} \subset (0, 1)$  with  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Then,  $\{x_n\}$  converges strongly to  $P_F x$ , where  $P_F$  is a sunny nonexpansive retraction of  $C$  onto  $F$ .*

## 5 Strong convergence theorem of Halpern's type

In this section, we prove two strong convergence theorems of Halpern's type for a countable family of nonexpansive mappings in a Banach space.

**Theorem 5.1 ([48]).** *Let  $E$  be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let  $C$  be a nonempty closed convex subset of  $E$  which has the fixed point property for nonexpansive mappings. Let  $T$  be a nonexpansive mapping of  $C$  into itself and let  $\{T_n\}$  be a family of nonexpansive mappings of  $C$  into itself which satisfy  $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ . Further, suppose that  $\{T_n\}$  satisfies the condition (I) with  $T$  and the condition (II). Let  $\{x_n\}$  be a sequence in  $C$  as follows:  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T_n x_n, \quad n = 1, 2, 3, \dots,$$

*where  $\{\alpha_n\} \subset [0, 1)$  satisfies  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $f$  is a contraction of  $C$  into itself. Then,  $\{x_n\}$  converges strongly to  $u \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ , where  $u = Pf(u)$  and  $P$  is a sunny nonexpansive retraction of  $C$  onto  $F(T)$ .*

Using Theorems 3.5 and 5.1, we obtain the following result:

**Theorem 5.2 ([48]).** *Let  $E$  be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Let  $C$  be a nonempty closed convex subset of  $E$  which has the fixed point property for nonexpansive mappings and let  $T : C \rightarrow C$  be a nonexpansive mapping such that  $F(T)$  is nonempty and let  $f$  be a contraction of  $C$  into itself. Define a sequence  $\{x_n\}$  of  $C$  as follows:  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)((1 - \beta_n)x_n + \beta_n T x_n) \quad \text{for all } n \in \mathbb{N},$$

*where  $\{\alpha_n\} \subset (0, 1)$  and  $\{\beta_n\} \subset (0, 1)$  satisfy the following conditions:*

$$\alpha_n \rightarrow 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < a \leq \beta_n \leq b < 1.$$

*Then, the sequence  $\{x_n\}$  converges strongly to a fixed point of  $T$ .*



Theorem 5.2 improves and extends Suzuki's result [42]. Using Theorems 3.7 and 5.1, we also obtain the following result which was proved by Takahashi [47].

**Theorem 5.3 ([47]).** *Let  $E$  be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let  $C$  be a nonempty closed convex subset of  $E$  which has the fixed point property for nonexpansive mappings. Let  $A \subset E \times E$  be an accretive operator with  $A^{-1}0 \neq \emptyset$  satisfying*

$$\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA),$$

where  $\overline{D(A)}$  is the closure of  $D(A)$  and let  $f$  be a contraction of  $C$  into itself. Let  $\{x_n\}$  be a sequence of  $C$  defined by  $x_1 = x \in C$  and

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{t_n} x_n \quad \text{for all } n \in \mathbb{N},$$

where  $\{\alpha_n\} \subset (0, 1)$  and  $\{t_n\} \subset (0, \infty)$  satisfy the following conditions:

$$\alpha_n \rightarrow 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad t_n \rightarrow \infty.$$

Then, the sequence  $\{x_n\}$  converges strongly to  $u \in A^{-1}0$ , where  $u = Pf(u)$  and  $P$  is a sunny nonexpansive retraction of  $C$  onto  $A^{-1}0$ .

**Theorem 5.4 ([24]).** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $\{T_n\}$  and  $T$  be families of nonexpansive mappings of  $C$  into itself which satisfy  $\emptyset \neq F(T) = \bigcap_{n=1}^{\infty} F(T_n)$ . Further, suppose that  $\{T_n\}$  satisfies the condition (I) with  $T$  and the condition (III). Let  $\{x_n\}$  be a sequence generated as follows:  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T_n(\beta_n x + (1 - \beta_n) x_n) \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $\{\beta_n\} \subset [0, 1)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$  and  $\prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = 0$ . If  $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ , then  $\{x_n\}$  converges strongly to  $P_{F(T)}x$ , where  $P_{F(T)}$  is a sunny nonexpansive retraction of  $C$  onto  $F(T)$ .

Using Proposition 3.1 and Theorem 5.4, we obtain the following theorem:

**Theorem 5.5.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $T$  be a nonexpansive mapping of  $C$  into itself with  $F(T) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated as follows:  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T(\beta_n x + (1 - \beta_n) x_n) \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $\{\beta_n\} \subset [0, 1)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = 0$  and  $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$ . Then,  $\{x_n\}$  converges strongly to  $P_{F(T)}x$ , where  $P_{F(T)}$  is a sunny nonexpansive retraction of  $C$  onto  $F(T)$ .

We have the following result [17] for nonexpansive mappings by Theorems 3.2 and 5.4.

**Theorem 5.6.** *Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $S$  and  $T$  be nonexpansive mappings of  $C$  into itself with  $F(S) \cap F(T) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated as follows:  $x_1 = x \in C$  and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)(\gamma_n S + (1 - \gamma_n)T)(\beta_n x + (1 - \beta_n) x_n) \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $\{\beta_n\} \subset [0, 1)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = 0$  and  $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$  and  $\{\gamma_n\} \subset [a, b]$  for some  $a, b \in (0, 1)$  with  $a \leq b$  satisfies  $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$ . Then,  $\{x_n\}$  converges strongly to  $P_{F(S) \cap F(T)}x$ , where  $P_{F(S) \cap F(T)}$  is a sunny nonexpansive retraction of  $C$  onto  $F(S) \cap F(T)$ .

We have the following result [21] for accretive operators from Theorems 3.7 and 5.4.

**Theorem 5.7.** Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable and let  $A \subset E \times E$  be an accretive operator with  $\overline{D(A)} \subset C \subset \cap_{\lambda > 0} R(I + \lambda A)$  and  $A^{-1}0 \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated as follows:  $x_1 = x \in C$  and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{\lambda_n}(\beta_n x + (1 - \beta_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $\{\beta_n\} \subset [0, 1)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$ ,  $\prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \beta_n) = 0$  and  $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n+1}| + |\beta_n - \beta_{n+1}|) < \infty$  and  $\{\lambda_n\} \subset (0, \infty)$  satisfies  $\liminf_{n \rightarrow \infty} \lambda_n > 0$  and  $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ . Then,  $\{x_n\}$  converges strongly to  $P_{A^{-1}0}x$ , where  $P_{A^{-1}0}$  is a sunny nonexpansive retraction of  $C$  onto  $A^{-1}0$ .

We get the following result [34] for  $W$ -mappings by Theorems 3.8 and 5.4.

**Theorem 5.8.** Let  $C$  be a nonempty closed convex subset of a uniformly convex Banach space  $E$  whose norm is uniformly Gâteaux differentiable. Let  $S_1, S_2, \dots$  be infinite nonexpansive mappings of  $C$  into itself with  $F := \cap_{n=1}^{\infty} F(S_n) \neq \emptyset$  and let  $\beta_1, \beta_2, \dots$  be real numbers with  $0 < \beta_i \leq b < 1$  for every  $i \in \mathbb{N}$  for some  $b \in (0, 1)$ . Let  $W_n$  be the  $W$ -mapping generated by  $S_n, S_{n-1}, \dots, S_1$  and  $\beta_n, \beta_{n-1}, \dots, \beta_1$  for every  $n \in \mathbb{N}$ . Let  $\{x_n\}$  be a sequence generated as follows:  $x_1 = x \in C$  and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) W_n(\gamma_n x + (1 - \gamma_n)x_n) \quad (\forall n \in \mathbb{N}),$$

where  $\{\alpha_n\} \subset [0, 1)$  and  $\{\gamma_n\} \subset [0, 1)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \gamma_n = 0$ ,  $\prod_{n=1}^{\infty} (1 - \alpha_n)(1 - \gamma_n) = 0$  and  $\sum_{n=1}^{\infty} (|\alpha_n - \alpha_{n+1}| + |\gamma_n - \gamma_{n+1}|) < \infty$ . Then,  $\{x_n\}$  converges strongly to  $P_F x$ , where  $P_F$  is a sunny nonexpansive retraction of  $C$  onto  $F$ .

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